Abstract

for the dissertation for the degree of Doctor of Philosophy (PhD)

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Linear and Nonlinear Inverse Problems for Evolutionary Equations with Degeneracy

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Relevance of the Topic: Inverse problems for differential equations play an important role in various fields of science and engineering, such as physics, engineering, biology, and medicine. They allow for the recovery of unknown parameters or initial conditions of a system based on observations of its behavior. Linear and nonlinear inverse problems for degenerate parabolic equations are of particular interest, as these equations describe many important processes, including heat conduction, diffusion, and the dynamics of various media.

Parabolic equations, especially in their degenerate form, are characterized by features such as changes in the type of equation depending on time and space, making their study particularly challenging and relevant. Degenerate parabolic equations arise, for example, when describing heat conduction processes in media with phase transitions or in heterogeneous materials where thermal conductivity may approach zero.

Studying the solvability of inverse problems for such equations involves developing methods for recovering the coefficients of the equation, initial and boundary conditions from given data. These problems are usually formulated as optimization or extremum problems, requiring the application of methods from functional analysis, operator theory, and numerical methods.

The main goal of this dissertation is to investigate the solvability of linear and nonlinear inverse problems for degenerate parabolic equations. This includes:

- Analyzing existing methods for solving inverse problems for degenerate parabolic equations and generalizing them to nonlinear cases.

- Developing new methods and algorithms for solving such problems, taking into account the specifics of degenerate equations.

- Conducting a theoretical analysis of the solvability of the problems, including proving existence and uniqueness theorems for solutions.

- Applying the developed methods to practical problems, such as recovering heat conductivity coefficients in heterogeneous media or identifying heat sources in complex systems.

Thus, this work aims to provide an in-depth study and development of methods for solving inverse problems for degenerate parabolic equations, which has significant theoretical and practical importance for various fields of science and engineering.

In inverse problems, the presence of additional unknown functions requires that, in addition to the boundary conditions natural for a given class of differential equations, some additional conditions—overdetermination conditions—are also specified. In this work, overdetermination conditions known in the literature as integral overdetermination conditions will be used. Inverse coefficient problems, both linear and nonlinear, with integral overdetermination conditions are quite well-studied for classical (elliptic, parabolic, and hyperbolic) as well as non-classical differential equations. However, for degenerate parabolic equations with respect to the time variable, inverse coefficient problems with integral overdetermination have not been studied previously.

Main Objective and Novelty

The main objective of the research is to address the solvability of both linear and nonlinear inverse problems for evolutionary equations with degeneration.

Research tasks:

- To prove the solvability of the linear inverse problem of determining time coefficients for a degenerate parabolic equation with changing evolution direction.

- To prove the solvability of the nonlinear inverse problem of determining time coefficients for a degenerate parabolic equation with changing evolution direction.

- To prove the solvability of the linear inverse problem of determining spatial coefficients for a degenerate parabolic equation with changing evolution direction.

- To prove the solvability of the nonlinear inverse problem of determining spatial coefficients for a degenerate parabolic equation with changing evolution direction.

- To prove the uniqueness of solutions to the nonlinear inverse problem of determining spatial coefficients for a degenerate parabolic equation with changing evolution direction.

- To prove the solvability of the nonlinear inverse problem for a strongly degenerate parabolic equation.

- To prove the uniqueness of solutions to the inverse problem for a strongly degenerate parabolic equation.

Object of the study: Linear and nonlinear inverse problems for evolutionary equations with degeneration.

Subject of the study: Solvability of coefficient inverse problems in various settings for evolutionary equations with degeneration.

Research Methodology

This dissertation uses methods from the general theory of partial differential equations, functional and mathematical analysis, as well as embedding theorems. The technique for proving the existence and uniqueness of regular solutions for the posed problems is based on transforming the original inverse problem into a new direct initial-boundary value problem for a corresponding integro-differential equation. Various methods, such as the method of continuation by parameter, the method of cutting functions, the method of a priori estimates, and the method of regularization, are used to establish the existence of solutions to the direct boundary (or initial-boundary) problem.

From the existence of a solution to the direct problem, a conclusion about the existence of a solution to the inverse problem can be drawn. This is because the solution to the direct problem is a necessary condition for the existence of a solution to the inverse problem, as the inverse problem involves recovering the parameters of the equation based on some known data about its behavior. If a solution to the direct problem exists, it means that data about the behavior of the equation is already known, and one can attempt to recover its parameters, which is the task of the inverse problem. However, this does not guarantee the uniqueness or stability of the solution to the inverse problem.

Thus, the obtained results on the existence of solutions to direct problems form a necessary foundation for considering inverse problems. They establish the basic possibility of recovering the parameters of the equation from known data about its behavior. This links the theory of inverse problems with the theory of direct problems and highlights the importance of understanding both aspects when

solving such problems.

Scientific Novelty

The solvability of inverse coefficient problems for degenerate differential equations, considered in §§1.1-1.2, §§2.1-2.2, and §§3.1, §§4.1, has not been previously investigated, and the theorems on solvability obtained in this work have independent value.

Theoretical and Practical Significance

The results of the dissertation are theoretical in nature. It uses the method of slices for a nonlinear equation, the method of transforming the inverse problem into a loaded integro-differential equation, the method of regularization, a priori estimates, and embedding theorems to prove the existence and uniqueness of solutions to the inverse problem for a degenerate partial differential equation.

The practical significance of this work is determined by the fact that the research results have practical relevance for solving problems in various fields, such as mathematical modeling of heat transfer processes, designing technical devices, medical diagnostics, etc.

Points to be Defended

1) Solvability of the linear inverse problem of determining time coefficients for a degenerate parabolic equation with changing evolution direction.

2) Solvability of the nonlinear inverse problem of determining time coefficients for a degenerate parabolic equation with changing evolution direction.

3) Solvability of the linear inverse problem of determining spatial coefficients for a degenerate parabolic equation with changing evolution direction.

4) Solvability of the nonlinear inverse problem of determining spatial coefficients for a degenerate parabolic equation with changing evolution direction.

5) Theorems of uniqueness of solutions to nonlinear inverse problems of determining spatial coefficients for a degenerate parabolic equation with changing evolution direction.

6) Solvability of the nonlinear inverse problem for a strongly degenerate parabolic equation.

7) Theorems of uniqueness of solutions to inverse problems for a strongly degenerate parabolic equation.

Reliability and Justification

The reliability and justification of the conducted research are ensured by the constructive and systematic use of standard methods for partial differential equations. This is confirmed by publications presented to the Committee for Control in the Field of Education and Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan and supported by conference materials.

Publications

The main results on the topic of the dissertation have been published in 7 articles and 6 abstracts.

Structure of the Dissertation

The dissertation consists of an introduction, four chapters, a conclusion, and a list of references.